

A Method for Enforcing the Solenoidal Condition on Magnetic Field in Numerical Calculations*

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A method is proposed for enforcing the solenoidal condition $\nabla \cdot \mathbf{B} = 0$ on the magnetic field \mathbf{B} at all times during a transient numerical calculation. The method is based on the observation that Faraday's law for the time evolution of \mathbf{B} can be cast into an alternative equivalent form which closely resembles the equations of incompressible hydrodynamics. In this alternative formulation, the solenoidal condition appears as a constraint that applies for all time, rather than just as an initial condition. The solenoidal condition may therefore be preserved during a transient calculation simply by basing the numerical scheme on the alternative formulation instead of the usual one. A possible numerical scheme is suggested which is analogous to the MAC method for incompressible fluid flow.

The transient evolution of the magnetic field is governed by Faraday's law,

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}. \quad (1)$$

Since $\nabla \cdot (\nabla \times \mathbf{E}) = 0$, Eq. (1) implies that $\nabla \cdot \mathbf{B}$ is independent of time. The solenoidal condition

$$\nabla \cdot \mathbf{B} = 0 \quad (2)$$

is therefore automatically satisfied for all time if it is satisfied initially. Thus Eq. (2) is not a *dynamical* condition on \mathbf{B} ; it should rather be regarded as a constraint on the initial conditions. That is, Eq. (2) does not belong to the system of evolution equations; the fact that it is satisfied for all time is a *consequence* of the evolution equation (1) and the initial conditions [1].

This situation gives rise to a difficulty in numerical calculations. The problem is that the numerical scheme used to approximate Eq. (1) will not in general preserve the differential property that $\nabla \cdot \mathbf{B}$ is rigorously independent of time. Thus, even though $\nabla \cdot \mathbf{B}$ is zero initially, it may slowly drift away from zero as a result of numerical discretization errors. Since Eq. (2) is not one of the evolution equations, these equations have no tendency to restore $\nabla \cdot \mathbf{B}$ to zero. Moreover, a numerical

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approximation to Eq. (2) cannot be added to the scheme because \mathbf{B} is already completely determined by the numerical approximation to Eq. (1).

It will be sufficient for our purposes to illustrate these considerations within the context of a finite-difference numerical scheme. A simple first-order temporal difference approximation to Eq. (1) is

$$\mathbf{B}^{n+1} = \mathbf{B}^n - c\Delta t \langle \nabla \times \rangle \mathbf{E}, \quad (3)$$

where $\langle \mathcal{O} \rangle$ denotes the spatial difference operator used to approximate the differential operator \mathcal{O} . The time level at which \mathbf{E} is evaluated is immaterial here. It follows from Eq. (3) that

$$\langle \nabla \cdot \rangle \mathbf{B}^{n+1} = \langle \nabla \cdot \rangle \mathbf{B}^n - c\Delta t \langle \nabla \cdot \rangle \langle \nabla \times \rangle \mathbf{E}. \quad (4)$$

The difficulty in question now manifests itself in the fact that, unlike their differential counterparts, the difference operators $\langle \nabla \cdot \rangle$ and $\langle \nabla \times \rangle$ will not in general have the property that $\langle \nabla \cdot \rangle \langle \nabla \times \rangle$ is identically zero. Even if $\langle \nabla \cdot \rangle \mathbf{B}^n$ is zero, $\langle \nabla \cdot \rangle \mathbf{B}^{n+1}$ will then differ slightly from zero, and the discrepancy may be expected to slowly become worse as time goes on. One would like to be able to impose the condition $\langle \nabla \cdot \rangle \mathbf{B}^{n+1} = 0$ as a constraint on the solution, but there is no freedom to do so as \mathbf{B}^{n+1} is already determined by Eq. (3).

Various types of unphysical behavior are found to accompany violations of the solenoidal condition on \mathbf{B} . For example, Brackbill and Barnes [2] have shown that artificial fluid velocities parallel to \mathbf{B} can result from a nonzero $\langle \nabla \cdot \rangle \mathbf{B}$ in numerical magnetohydrodynamics calculations. They find that these velocity errors can be greatly reduced by differencing a nonconservative form of the fluid momentum equation, even though no attempt is made to reduce or eliminate $\langle \nabla \cdot \rangle \mathbf{B}$ itself. This finding suggests that, in particular cases at least, it may be possible to tolerate a nonzero $\langle \nabla \cdot \rangle \mathbf{B}$ if ways can be found to control its principal unphysical consequences. Our concern here, however, is with the basic solenoidal condition itself, and not with the secondary effects that may attend its violation in various circumstances. It should be noted that the accumulation of numerical errors in $\langle \nabla \cdot \rangle \mathbf{B}$ is a problem of general relevance to transient calculations in electrodynamics and plasma physics, as well as magnetohydrodynamics.

There are at least two obvious remedies for this problem. The first is to introduce the vector potential \mathbf{A} (from which \mathbf{B} is obtained by $\mathbf{B} = \nabla \times \mathbf{A}$), and to replace Eq. (1) by the evolution equation for \mathbf{A} . The second is to construct the difference operators $\langle \nabla \cdot \rangle$ and $\langle \nabla \times \rangle$ so that $\langle \nabla \cdot \rangle \langle \nabla \times \rangle$ *does* vanish identically. The first remedy is quite commonly used, especially in two dimensions where it has the advantage that \mathbf{A} becomes essentially a scalar function. Nevertheless, many workers retain a decided preference for the "primitive" variable \mathbf{B} . The second remedy would probably be the method of choice if it could be easily realized, but the construction of difference operators with the desired property is difficult and cumbersome, especially in a generalized non-rectangular spatial mesh. Our purpose here is to propose a third

remedy, in which the primitive variable \mathbf{B} is retained with no restrictions on the form of the difference operators $\langle \nabla \cdot \rangle$ and $\langle \nabla \times \rangle$.

Consider the following system of equations:

$$\nabla \cdot \mathbf{B} = 0, \quad (5)$$

$$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E} + \nabla \phi, \quad (6)$$

where Eq. (5) is now required to be satisfied at all times, not merely initially, and ϕ is a scalar function. (Indeed, it is clear that Eq. (5) *must* be applied at all times to determine the additional dependent variable ϕ ; otherwise there would be more unknowns than equations.) Since Eqs. (5) and (6) do not involve time derivatives of ϕ , no initial conditions on ϕ are needed. However, boundary conditions on ϕ must be specified on the surface bounding the region of interest. There is a certain amount of freedom in the choice of these boundary conditions. For our purposes, it will be sufficient to require that one of the following choices be adopted: (a) ϕ is a constant or (b) the normal derivative of ϕ vanishes.

We now proceed to show that Eqs. (5) and (6) are in fact equivalent to Eq. (1) with the solenoidal constraint imposed on the initial condition only. If the latter condition is understood, Eq. (5) may be replaced by

$$\frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}) = 0. \quad (7)$$

The divergence of Eq. (6) combines with Eq. (7) to yield

$$\nabla^2 \phi = 0, \quad (8)$$

and we may therefore replace Eq. (7) with Eq. (8). But the solution to Eq. (8), subject to either boundary conditions (a) or (b), is simply that ϕ is a constant, independent of position. Therefore $\nabla \phi$ vanishes and Eq. (6) reduces to Eq. (1), thereby demonstrating the equivalence.

Notice that the elimination of ϕ to regain Eq. (1) is inherently accompanied by the removal of Eq. (5) from the equation system (except as an initial condition). It is the presence of the term $\nabla \phi$ in Eq. (6) that gives us the freedom to bring in Eq. (5) as a dynamical condition that applies for all time. If we wish to retain this condition, as we do for numerical purposes, we must retain $\nabla \phi$ in Eq. (6) as well.

We note that Eqs. (5) and (6) are closely analogous to the equations of incompressible hydrodynamics: \mathbf{B} corresponds to the fluid velocity, ϕ to the pressure, and Eq. (5) to the incompressibility condition. This analogy suggests that a useful numerical scheme for Eqs. (5) and (6) might be patterned after the MAC method for incompressible flow [3, 4]. The resulting scheme, which replaces Eq. (3), is given by

$$\langle \nabla \cdot \rangle \mathbf{B}^{n+1} = 0, \quad (9)$$

$$\mathbf{B}^{n+1} = \mathbf{B}^n - c \Delta t \langle \nabla \times \rangle \mathbf{E} + \Delta t \langle \nabla \cdot \rangle \phi^{n+1}. \quad (10)$$

This is an implicit system of equations for \mathbf{B}^{n+1} and ϕ^{n+1} . The analogous system in incompressible flow has usually been solved by a pointwise iterative procedure, but other methods are of course also available.

If \mathbf{E} is evaluated at the old time level n , a single Poisson-like equation for ϕ^{n+1} may be obtained by combining Eqs. (9) and (10):

$$\langle \nabla^2 \rangle \phi^{n+1} = c \langle \nabla \cdot \rangle \langle \nabla \times \rangle \mathbf{E} - \langle \nabla \cdot \rangle \mathbf{B}^n / \Delta t, \quad (11)$$

where $\langle \nabla^2 \rangle = \langle \nabla \cdot \rangle \langle \nabla \cdot \rangle$, and the last term allows for the possibility that $\langle \nabla \cdot \rangle \mathbf{B}^n$ may differ slightly from zero due to a finite iteration convergence criterion. Inclusion of this term can greatly improve computational efficiency [5]. Of course, there is no *requirement* that one actually form and solve Eq. (11); in many cases it may be preferable to solve Eqs. (9) and (10) simultaneously, as is frequently done in the MAC method [4]. Different procedures may be expected to be advantageous in different contexts. In magnetohydrodynamics, for example, \mathbf{E} is expressed in terms of \mathbf{B} by means of Ohm's law (and Ampere's law if the electrical conductivity is finite). If the resulting expression is evaluated at the new time level $n + 1$, as is frequently desirable [6, 7], then even the straightforward scheme of Eq. (3) becomes implicit in \mathbf{B}^{n+1} . One might then hope that the additional implicitness of Eqs. (9) and (10) could be incorporated into existing schemes of this type with relatively small increases in complexity and computation time.

Of course, the scheme of Eqs. (9) and (10), like that of Eq. (3), is also subject to numerical errors arising from the fact that $\langle \nabla \cdot \rangle \langle \nabla \times \rangle$ is not identically zero. The advantage of Eqs. (9) and (10) is that such errors are not allowed to manifest themselves as an accumulating violation of the solenoidal condition. They will instead appear in the form of a nonuniform ϕ , as a result of the source term $c \langle \nabla \cdot \rangle \langle \nabla \times \rangle \mathbf{E}$ in Eq. (11). The situation is somewhat analogous to the use of conservative difference schemes for the mass continuity equation. Such schemes will in general yield numerical errors of the same order as nonconservative schemes, but one is assured that these errors will not manifest themselves as violations of mass conservation.

Finally, we remark that the present approach provides a rational justification for the obvious but seemingly ad hoc procedure of simply *discarding* the irrotational part of \mathbf{B}^{n+1} , as given by Eq. (3), after each time step. The natural way to implement this procedure is that mentioned in passing by Brackbill and Barnes [2]: Solve for a potential function ψ that satisfies the Poisson equation $\langle \nabla^2 \rangle \psi = -\langle \nabla \cdot \rangle \mathbf{B}^{n+1}$, with \mathbf{B}^{n+1} obtained from Eq. (3); then replace \mathbf{B}^{n+1} by $\mathbf{B}^{n+1} + \langle \nabla \cdot \rangle \psi$. (This is a standard way of extracting the solenoidal or transverse part of an arbitrary vector field [8]; it is sometimes referred to as "divergence cleaning.") Comparison with Eq. (11) shows that $\psi = \Delta t \phi^{n+1}$ (provided ψ is required to satisfy the same boundary conditions as ϕ), from which it readily follows that the resulting solenoidal \mathbf{B}^{n+1} is precisely the

same as that obtained in the present method. Thus the procedure in question, which at first may appear quite unjustified, is in fact a well-defined, natural, and consistent difference approximation to the governing differential equations in the form of Eqs. (5) and (6). It should be noted, however, that the boundary conditions on ψ in this procedure may not be chosen arbitrarily; they must be such that the solution to $\langle \nabla^2 \rangle \psi = 0$ is $\psi = \text{constant}$. Otherwise the \mathbf{B}^{n+1} obtained from Eq. (3) would be modified even if it were already solenoidal, and the scheme would no longer be consistent.

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